Compositionality and inferential roles of logical constants

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Abstract. Discussions on the compositionality of inferential roles concentrate on extralogical vocabulary. However, there are nontrivial problems concerning the compositionality of sentences formed by the standard constants of propositional logic. For example, is the inferential role of $A \wedge B$ uniquely determined by those of A and B? And how is it determined? This paper investigates such questions. We also show that these issues raise matters of more significance than may *prima facie* appear.

Keywords. Inferentialism; logical constants; compositionality

1. Inferentialism and compositionality

Some semantic theories suggest that the meanings of linguistic expressions are to be understood as the ways the expressions are put to use by competent speakers. And inferentialism suggests that we should focus our interest on the inferences which sentences containing these expressions undergo. Elsewhere (Peregrin, 2014) I suggest that we should distinguish between what can be called the *causal* and the *normative* versions of inferentialism and consequently of inferential roles. The causal variety of inferential role semantics sees the inferential roles as derived from what we *de facto* do with the expressions, which inferences we actually draw using them. The normative variety, in contrast to this, derives the roles from what we do with them *de jure* - i.e. from the *rules* governing the expressions. Here, I will concentrate on the normative version, which is essentially due to Brandom (1994).

One of the problems concerning any theory of meaning is that it is commonly held that it should be compositional[2] - and here the semantics of inferential roles gets challenged (see, e.g., Fodor & Lepore, 2001). I have defended it (Peregrin, 2009), but the whole discussion suffers from the fact that it is not clear what exactly the inferential roles are. Therefore, here I will try to work with exactly specified explications.

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[2] Not everybody subscribes to this. Importantly, Brandom (2008, §5.6) holds that compositionality can be replaced by recursive projectibility. But see Fermüller (2010).

Also, I will not here continue the discussion which generally concerns empirical – and in any case extralogical – expressions. Instead, I will concentrate on a more fundamental problem: I will investigate how do the inferential roles of standard logical constants support compositionality, in the sense that the inferential role of, say, a conjunction is determined by – or is computable from – the roles of its conjuncts. Again, to assess this problem properly, we need to know what exactly the inferential roles are.

So what is the inferential role of an expression? Basically, it is the expression's contribution to the inferential behavior of the sentences containing the expression. Therefore, I introduced the term *inferential potential* for the summary of the sentences which are inferable from a sentence and those it is inferable from (Peregrin, 2014, §3.3). Thus, if A is a sentence, the inferential potential of the sentence A can be considered as consisting of two parts, the first of them capturing its role of being a premise,

$$\|A\|^{\rightarrow} = \{\langle X,B\rangle \mid X,A \vdash B\},$$

and the second one its role of being a conclusion,

$$\|A\|^{\leftarrow} = \{X \mid X \vdash A\}.$$

(Here A, B are sentences and X is a set of sentences.) Thus $\|A\|^{\leftarrow}$ is the set of all sets of sentences from which A is inferable; while $\|A\|^{\rightarrow}$ contains all sets of sentences inferable from A together with all kinds of collateral premises.

Let us start in a framework where all the structural rules[3] hold. Given this, $\|A\|^{\leftarrow}$ and $\|A\|^{\rightarrow}$ are not independent; in particular

Claim 1. $\|A\|^{\leftarrow} = \|B\|^{\leftarrow}$ iff $\|A\|^{\rightarrow} = \|B\|^{\rightarrow}$.

Proof: Assume $\|A\|^{\leftarrow} = \|B\|^{\leftarrow}$. Let $\langle X,C\rangle \in \|A\|^{\rightarrow}$, i.e. $X,A \vdash C$. Then as $\{B\} \in \|B\|^{\leftarrow}$ (by REF), $\{B\} \in \|A\|^{\leftarrow}$, i.e. $B \vdash A$, we can infer $X,A \vdash C$ (by CUT), i.e. $\langle X,C\rangle \in \|B\|^{\rightarrow}$. Hence if $\|A\|^{\rightarrow} \subseteq \|B\|^{\rightarrow}$. The inverse inclusion is analogous, so $\|A\|^{\rightarrow} = \|B\|^{\rightarrow}$.

[3] The structural rules are the following:

(REF)	$A \vdash A$
(EXT)	if $X,Y \vdash A$ then $X,B,Y \vdash A$
(CON)	if $X,A,A,Y \vdash B$ then $X,A,Y \vdash B$
(PERM)	if $X,A,B,Y \vdash C$ then $X,B,A,Y \vdash C$
(CUT)	if $X,A,Y \vdash B$ and $Z \vdash A$ then $X,Z,Y \vdash B$

The rules (REF), (CON) and (PERM) are assumed throughout the whole paper, thus letting us work with *sets* (rather than sequences) of premises.

Now assume $\|A\|^{\rightarrow} = \|B\|^{\rightarrow}$. As $<\emptyset,B>\in\|B\|^{\rightarrow}$ (by REF), $<\emptyset,B>\in\|A\|^{\rightarrow}$ i.e. A $\vdash$ B. But if $X\in\|A\|^{\leftarrow}$, this implies $X\in\|B\|^{\leftarrow}$ (by CUT), so $\|A\|^{\leftarrow}\subseteq\|B\|^{\leftarrow}$. The inverse inclusion is analogous, so $\|A\|^{\leftarrow} = \|B\|^{\leftarrow}$. □

It follows that, given the structural rules, the inferential potential of a sentence can be characterized by either one of its components alone, $\|A\|^{\leftarrow}$ or $\|A\|^{\rightarrow}$.

For some simple languages, the inferential role of a sentence can be identified directly with its inferential potential; in more complex languages this is not possible. The reason is that a sentence may be a part of more complex sentences and sentences with identical potentials may not be intersubstitutive saving the inferential potential of the complex sentences. (This holds especially for languages that are *hyperintensional*, perhaps containing propositional attitude reports - see Cresswell (1975).) Here we will consider exclusively languages for which it is possible to identify a sentence's inferential role with its inferential potential, and hence our definition of the inferential potential may serve us directly as a definition of the inferential role:

(IR) $\|A\| = <\|A\|^{\leftarrow},\|A\|^{\rightarrow}>$, where $\|A\|^{\leftarrow} = \{X \mid X \vdash A\}$ and $\|A\|^{\rightarrow} = \{<X,B> \mid X,A \vdash B\}$

2. Preliminaries

What, in general, is compositionality? If we denote the meaning of an expression E as $\|E\|$, then the principle of compositionality says that for every syntactic rule R there is a function R* such that if R combines expressions E_1, ..., E_n into a complex expression R(E_1,...,E_n), then R* combines their meanings into the meaning of the complex (Dever, 1999; Hodges, 2001; Janssen, 2001; Peregrin, 2005):

(PC) $\|R(E_1,...,E_n)\| = R^*(\|E_1\|,...,\|E_n\|)$.

Applied to propositional logic, we have a rule CON that combines two sentences and a propositional connective (usually $\wedge$, $\vee$, or $\rightarrow$) into a sentence, hence there must be a function CON* such that

$\|CON(S_1,C,S_2)\| = CON^*(\|S_1\|, \|C\|, \|S_2\|)$.

In particular, for an individual connective, say $\wedge$, there must be a function $CON_{\wedge}^*$ such that

$\|S_1\wedge S_2\| = CON_{\wedge}^*(\|S_1\|,\|S_2\|)$.

Indeed, $CON_{\wedge}^*(\|S_1\|,\|S_2\|) = CON^*(\|S_1\|, \|\wedge\|, \|S_2\|)$.. (If $\|\wedge\|$ is taken to be the classical truth function, and $\|S_1\|$, $\|S_2\|$ and $\|S_1\wedge S_2\|$ the truth values of the sentences, then $CON_{\wedge}^* = \|\wedge\|$.)

Related to the principle of compositionality there is what can be called the principle of intersubstitutivity of synonyms:

(PIS) if $\|E_i\| = \|E_i'\|$, then $\|R(E_1,\ldots,E_i,\ldots,E_n)\| = \|R(E_1,\ldots,E_i',\ldots,E_n)\|$.

Claim 2. (PC) iff (PIS).

Proof. Assume (PC), i.e. $\|R(E_1,\ldots,E_i,\ldots,E_n)\| = R^*(\|E_1\|,\ldots,\|E_i\|,\ldots,\|E_n\|)$. If $\|E_i\| = \|E_i'\|$, then $R^*(\|E_1\|,\ldots,\|E_i\|,\ldots,\|E_n\|) = R^*(\|E_1\|,\ldots,\|E_i'\|,\ldots,\|E_n\|) = \|R(E_1,\ldots,E_i',\ldots,E_n)\|$.

Now assume PIS. Then if $\|E_1\| = \|E_1'\|$, ..., $\|E_n\| = \|E_n'\|$, then $\|R(E_1,\ldots,E_n)\| = \|R(E_1',\ldots,E_n')\|$. □

Note that the fact that (PIS) entails (PC) depends on the broadest understanding of the concept of function. On this understanding, it is enough that every *n*-tuple from the domain uniquely determines an object from the range. However, this broad understanding of the concept of function guarantees merely that the arguments uniquely determine the value, not that we are always able to find it. Therefore, (PC) is often connected with some more transparent way of determination of the meaning of a whole by the meanings of its parts, such as the function application of the meaning of one of the parts to those of the rest of them

$$\|R(E_1,\ldots,E_{i-1},E_i,E_{i+1},\ldots,E_n)\| = \|E_i\|(\|E_1\|,\ldots,\|E_{i-1}\|,\|E_{i+1}\|,\ldots,\|E_n\|)$$

Hence it is not only important that there is a function but also that there is such a function which we can always apply to the arguments with the effect of getting the value.

3. Inferential roles w.r.t. single-conclusion inference

One might expect that compositionality of inferential roles at least for classical logic holds quite trivially; and perhaps that the inferential roles add up to each other quite transparently. For example, the fact that the conjunction of two sentences is assertable iff each of them is might seem to provide for a transparent compositionality of conjunction. But with inferential roles it is not so simple.

Let us stay, for a while, in the standard setting. We consider roles w.r.t. single conclusion inference assuming all the structural rules (thus, in effect, within the framework of Gentzenian natural deduction). Let us start with conjunction. The usual inferential pattern governing it is the following[4]:

($\wedge$E1) $A\wedge B \vdash A$

($\wedge$E2) $A\wedge B \vdash B$

($\wedge$I) $A,B \vdash A\wedge B$

[4] The whole system of rules of natural deduction can be found, e.g., in Prawitz (1965).

It is clear that ($\wedge$E1) and ($\wedge$E2) guarantee that

$$\|A\wedge B\|^{\leftarrow} \subseteq \|A\|^{\leftarrow}\cap\|B\|^{\leftarrow}$$

At the same time, ($\wedge$I) yields us

$$\|A\|^{\leftarrow}\cap\|B\|^{\leftarrow} \subseteq \|A\wedge B\|^{\leftarrow}$$

and hence we have

$$\|A\wedge B\|^{\leftarrow} = \|A\|^{\leftarrow}\cap\|B\|^{\leftarrow}.$$

Also, ($\wedge$E1) and ($\wedge$E2) yield us

$$\|A\|^{\rightarrow}\cup\|B\|^{\rightarrow} \subseteq \|A\wedge B\|^{\rightarrow}.$$

Thus we may be tempted to assume that ($\wedge$I) will yield us

$$?\ \|A\wedge B\|{\rightarrow}\subseteq \|A\|{\rightarrow}\cup\|B\|{\rightarrow}$$

and hence we would have

$$?\ \|A\wedge B\|{\rightarrow} = \|A\|{\rightarrow}\cup\|B\|{\rightarrow}$$

making the compositionality of the inferential role for conjunction trivial. Alas, this is *not* the case, for $\|A\wedge B\|^{\rightarrow}$is *not* necessarily a subset of $\|A\|^{\rightarrow}\cup\|B\|^{\rightarrow}$. The point is that there may be an X and C such that $X\cup\{A\wedge B\} \vdash C$, but neither $X\cup\{A\} \vdash C$, nor $X\cup\{B\} \vdash C$. (The conjunction *This is featherless and this is a biped* yields us *This is human*, though none of the conjuncts alone yields it.) Hence the situation is less trivial. However, in view of the fact that $X\cup\{A\wedge B\} \vdash C$ iff $X\cup\{A,B\} \vdash C$ it is the case that

$$\|A\wedge B\|^{\rightarrow} = \{\langle X,C\rangle \mid X\cup\{A,B\} \vdash C \}.$$

However, this is not yet what we need: we need a way to get $\|A\wedge B\|$ from $\|A\|$ and $\|B\|$, whereas what we have is how to get it from A and B.

Hence what we need is to replace the reference to A and B by the reference to $\|A\|$ and $\|B\|$. This is less difficult: we can replace A by any expression with the inferential role of A, for we need not distinguish between expressions with the same inferential role. Thus we need to define the set of all instances of the role $\|A\|$ (viz. the set of all expressions sharing the inferential role of A). This also is not difficult:

$$i(\|A\|) = \{B \mid \{B\}\in\|A\|^{\leftarrow} \text{ and } \{\langle\emptyset,B\rangle\}\in\|A\|^{\rightarrow}\}.$$

It is now easily seen that $A\in i(\|A\|)$ and if $B\in i(\|A\|)$, then $C\in i(\|A\|)$ iff $B \vdash C$ and $C \vdash B$ (which we will abbreviate to $B \dashv\vdash C$), i.e. if B and C are interinferable and hence inferentially equivalent.

Now we can write

$$\|A\wedge B\|^{\rightarrow} = \{\langle X,C\rangle \mid \exists D\exists E\text{: } D\in i(\|A\|) \text{ and } E\in i(\|B\|) \text{ and } X\cup\{D,E\} \vdash C\}.$$

Not trivial, but still perspicuous enough.

Now consider disjunction. In view of its duality with conjunction, we may expect that we can get the corresponding equations as some transposition of those for conjunction. And indeed this holds for $\|A\vee B\|^{\rightarrow}$, as

$$\|A\vee B\|^{\rightarrow} = \|A\|^{\rightarrow}\cap\|B\|^{\rightarrow}.$$

However, the situation with $\|A\vee B\|^{\leftarrow}$ is trickier. This, of course, is no surprise: while the introduction rules for disjunction are dual to the elimination rules for conjunction:

$(\vee I1)\ A \vdash A\vee B$,

$(\vee I2)\ B \vdash A\vee B$;

the eliminating rule cannot be based on the duality:

$(\vee E)\ [A]C, [B]C, A\vee B \vdash C$.

The duality with conjunction is unusable, for to transpose the equation for conjunction we would need to have more than one sentence on the right hand side of $\vdash$. But so far we are working with the single-conclusion version of $\vdash$.

4. Intersubstitutivity of interderivables

The difficulties we encountered in the previous section may make us wonder whether the inferential roles of logically complex sentences are compositional at all. But it is easy to show that they are. Remember that the principle of compositionality is entailed by the principle of intersubstitutivity of synonyms and hence it is enough to show that if $\|A\| = \|A'\|$ and $\|B\| = \|B'\|$, then $\|\neg A\| = \|\neg A'\|$ and $\|A\#B\| = \|A'\#B'\|$ for # being $\wedge$, $\vee$, and $\rightarrow$.

It is easy to see that $\|A\| = \|A'\|$ iff A and A' are interderivable, i.e. iff $A \dashv\vdash A'$. Indeed, if they are, then $X, A \vdash B$ iff $X, A' \vdash B$ and $X \vdash A$ iff $X \vdash A'$ (using CUT). Hence what we need to prove is[5]

Theorem 1. if $A \dashv\vdash A'$ and $B \dashv\vdash B'$, then $\neg A \dashv\vdash \neg A'$) and $A\#B \dashv\vdash A'\#B'$, for # being given $\wedge$, $\vee$, and $\rightarrow$.

Proof. We prove it for negation and implication only; the rest is easy.

As $\neg$-elimination gives us

[5] For proofs of this fact in the context of Hilbertian axiomatic systems (and predicate logic) see Kleene (1967, p. 122) or Shoenfield (1967, p. 34).

$$A, \neg A \vdash \bot,$$

we can derive (from this and $A' \vdash A$ by (CUT))

(*) $$A', \neg A \vdash \bot.$$

But as $\neg$-introduction gives us

$$[A']\bot \vdash \neg A'$$

using (*) as $[A']\bot$, we have

$$\neg A \vdash \neg A'.$$

Now $\rightarrow$-elimination gives us

$$A, A \rightarrow B \vdash B.$$

we have (from $A' \vdash A$ and $B \vdash B'$ by (CUT))

(**) $$A', A \rightarrow B \vdash B'.$$

But $\rightarrow$-introduction gives us

$$[A']B' \vdash A' \rightarrow B'.$$

Now if we take (**) as $[A']B'$, we have

$$A \rightarrow B \vdash A' \rightarrow B'. \square$$

Returning now to the definition of the inferential role of conjunction:

$$\|A \wedge B\|^{\rightarrow} = \{\langle X,C \rangle \mid \exists D \exists E\text{: } D \in i(\|A\|) \text{ and } E \in i(\|B\|) \text{ and } X \cup \{D,E\} \vdash C\}.$$

As $D \in i(\|A\|)$ iff $D \dashv\vdash A$ and $E \in i(\|B\|)$ iff $E \dashv\vdash B$, it is the case that $X \cup \{D,E\} \vdash C$ iff $X \cup \{A,B\} \vdash C$ (given CUT) and hence our definition holds iff

$$\|A \wedge B\|^{\rightarrow} = \{\langle X,Y \rangle \mid X,A,B \vdash Y)\}.$$

We saw that the framework of multiple-conclusion inference (sequent calculus) would enable us to handle disjunction as directly dual to conjunction. So let us move to a multiple-conclusion framework, where we can have a set of sentences not only on the left hand side of $\vdash$, but also on its right hand side. (After all, we know that this move can straightforwardly yield us logic with classical, truth-functional semantics[6].)

[6] Of course it is possible to get classical logic even in the natural deduction settings - e.g. by adding the *excluded middle* to the system of intuitionistic logic. However, as already Carnap noticed, this leads to a semantics that is not really truth-functional - see, e.g., Raatikainen (2008).

6. Multiple-conclusion inference

Modifying the definition of the inferential role for multiple-conclusion inference, we get

(IR*) $\|A\| = <\|A\|^{\leftarrow},\|A\|^{\rightarrow}>$,

$$\text{where } \|A\|^{\leftarrow} = \{<X,Y> \mid X \vdash Y\cup\{A\}\} \text{ and } \|A\|^{\rightarrow} = \{<X,Y> \mid X\cup\{A\} \vdash Y\}$$

and the set of instances for the inferential role is

$$i(\|A\|) = \{B \mid <B,\emptyset>\in\|A\|^{\leftarrow} \text{ and } <\emptyset,B>\in\|A\|^{\rightarrow}\}.$$

The problematic half of the (IR*) for conjunction then is

$$\|A\wedge B\|^{\rightarrow} = \{<X,Y> \mid \exists D\exists E\text{: } D\in i(\|A\|) \text{ and } E\in i(\|B\|) \text{ and } X\cup\{D,E\} \vdash Y\}.$$

and now we can produce the prescription for disjunction as its direct dual:

$$\|A\vee B\|^{\leftarrow} = \{<X,Y> \mid \exists D\exists E\text{: } D\in i(\|A\|) \text{ and } E\in i(\|B\|) \text{ and } X \vdash Y\cup\{D,E\}\}.$$

We can also produce a prescription for implication:

$$\|A\rightarrow B\|^{\rightarrow} = \|A\|^{\leftarrow}\cup\|B\|^{\rightarrow}.$$

$$\|A\rightarrow B\|^{\leftarrow} = \{<X,Y> \mid \exists D\exists E\text{: } D\in i(\|A\|) \text{ and } E\in i(\|B\|) \text{ and } X\cup\{D\} \vdash Y\cup\{E\}\}.$$

and also a very simple prescription for negation

$$\|\neg A\|^{\leftarrow} = \|A\|^{\rightarrow}$$

$$\|\neg A\|^{\rightarrow} = \|A\|^{\leftarrow}.$$

Given this, it is easy to see that $\|\neg\neg A\| = \|A\|$ and hence that the negation resulting from this definition is classical; which tallies with the common observation that the most direct way from multiple-conclusion inference to classical logic, whereas single conclusion inference leads us more directly to intuitionistic logic[7].

7. Going substructural

So far we have been basing our considerations on logic that complies with all the structural rules[8]. However, we can question the appropriateness of this, given Brandom's insistence that

[7] See Peregrin (2008).

[8] The structural rules for multiple-conclusion inference are as follows:

(REF)	$A \vdash A$	
(EXT)	if $X,Y \vdash Z$ then $X,A,Y \vdash Z$	if $X \vdash Y,Z$ then $X \vdash Y,A,Z$
(CON)	if $X,A,A,Y \vdash Z$ then $X,A,Y \vdash Z$	if $X \vdash Y,A,A,Z$ then $X \vdash Y,A,Z$

the material inferences forming the backbone of the inferential structure of any natural language are, by their very nature, non-monotonic, and hence do not support the rule (EXT) (also known as weakening). Hlobil & Brandom (2024, §3.1) argue that while (EXT) directly contradicts non-monotonicity, (EXT) is yielded by (CUT), given that we accept X $\vdash$Y whenever both X and Y contain the same sentence. Indeed, if this is the case, we have A,B,C $\vdash$ B and hence A $\vdash$ B yields us A,C $\vdash$ B.

Hence it may be prudent not to count with all the structural rules, especially not with (EXT) or (CUT). But can we make do without (CUT)? Return to the statement

$\|A\wedge B\|^{\leftarrow} \subseteq \|A\|^{\leftarrow}\cap\|B\|^{\leftarrow}$,

which we proclaimed a direct consequence of

($\wedge$E1) $A\wedge B \vdash A$

($\wedge$E2) $A\wedge B \vdash B$

Indeed, if $X\in\|A\wedge B\|^{\leftarrow}$, i.e. $X \vdash A\wedge B$, then ($\wedge$E1) yields us $X \vdash A$, and hence $X\in\|A\|^{\leftarrow}$. However, to get from $X \vdash A\wedge B$ and $A\wedge B \vdash A$ to $X \vdash A$ we need (CUT). Hence if we were to abandon (CUT), then our previous considerations would seem to be completely in pieces.

We can save ourselves, at least partially, by switching from the framework of natural deduction to that of sequent calculus. Here the elimination rules for $\wedge$ would turn into[9]

$$\frac{X \vdash A\wedge B}{X \vdash A}$$

Hence we get from $X \vdash A\wedge B$ to $X \vdash A$ directly, without the help of (CUT). Also we replace the structural rule (EXT) by the rule (CON), stipulating X $\vdash$ Y whenever both X and Y contain the same sentence. This saves at least some part of (EXT).

However, some parts of our previous considerations still *are* in pieces. For example, we assumed that two sentences have the same inferential role once they are interderivable, which, without (CUT), no longer holds. Thus, the definition of an instance

$i(\|A\|) = \{B \mid \{B\}\in\|A\|^{\leftarrow} \text{ and } \{<\emptyset,B>\}\in\|A\|^{\rightarrow}\}$.

is no longer usable. We must retreat to something like

(PERM) if $X,A,B,Y \vdash Z$ then $X,B,A,Y \vdash Z$ if $X \vdash Y,A,B,Z$ then $X \vdash Y,B,A,Z$

(CUT) if $X,A,Y \vdash Z$ and $U \vdash V,A,W$ then $X,U,Y \vdash V,Z,W$

[9] A system of rules for natural direction can also be found in Prawitz (1965). However a more inferentialism-friendly system is presented by Hlobil & Brandom (2024). As it consists of reversible rules, the following derivation can be obtained from the inversion of the right rule for conjunction.

$i(\|A\|) = \{B \mid \|B\| = \|A\|\}$.

But it is also possible to take a wholly different path.

8. Inferential roles, Kaplan-style

In his dissertation, Kaplan (2022) took a slightly different approach to inferential roles. To present his proposal we need some auxiliary definitions:

$$U^{\Upsilon} \equiv_{Def} \{\langle X,Y\rangle \mid \text{for every } \langle X',Y'\rangle \in U,\ X \cup X' \vdash Y \cup Y'\}$$

$$\langle X,Y\rangle \sqcup \langle X',Y'\rangle \equiv_{Def} \langle X \cup X', Y \cup Y'\rangle.$$

$$U \sqcup V \equiv_{Def} \{\langle X \cup X', Y \cup Y'\rangle \mid \langle X,Y\rangle \in U \text{ and } \langle X',Y'\rangle \in V\}.$$

Now Kaplan claims that

$$\|A \wedge B\|^{\rightarrow} = ((\|A\|^{\rightarrow})^{\Upsilon} \sqcup (\|B\|^{\rightarrow})^{\Upsilon})^{\Upsilon}.$$

This looks very different from our findings above; and it is quite intransparent. Let me show how we can reach the same result throwing some light on its nature. First, one more definition:

$$\langle X_1,Y_1\rangle \models \langle X_2,Y_2\rangle \equiv_{Def} \forall \langle X,Y\rangle: \text{if } X_1 \cup X \vdash Y_1 \cup Y \text{ then } X_2 \cup X \vdash Y_2 \cup Y.$$

Before we get to Kaplan's proposal, let us prove some auxiliary results.

Claim 3. If $\langle X_1,Y_1\rangle \models \langle X_2,Y_2\rangle$, then $\{\langle X,Y\rangle \mid X, X_1 \vdash Y, Y_1\} = \{\langle X,Y\rangle \mid X, X_1 \vdash Y, Y_1 \text{ and } X, X_2 \vdash Y, Y_2\}$.

Proof. That the first set is a subset of the second one is a direct consequence of the definition $\models$. That the second one is a subset of the first follows from the fact that the condition constitutive of the first set follows from that constitutive of the second one. □

Claim 4. $U^{\Upsilon\Upsilon} = \{\langle X,Y\rangle \mid U \models \langle X,Y\rangle\}$.

Proof. It is the case that $\langle X,Y\rangle \in U^{\Upsilon\Upsilon}$ iff $\forall \langle X_2,Y_2\rangle \in U^{\Upsilon}$: $X_2 \cup X \vdash Y_2 \cup Y$; hence $\langle X_2,Y_2\rangle \in U^{\Upsilon}$ iff $\forall \langle X_1,Y_1\rangle \in U$: $X_1 \cup X_2 \vdash Y_1 \cup Y_2$. Put together, $\langle X,Y\rangle \in U^{\Upsilon\Upsilon}$ iff $\forall \langle X_2,Y_2\rangle$: if $X_1 \cup X_2 \vdash Y_1 \cup Y_2$ for $\forall \langle X_1,Y_1\rangle \in U$, then $X \cup X_2 \vdash Y \cup Y_2$. □

Claim 5. $\|A\|^{\rightarrow} = \langle A,\varnothing\rangle^{\Upsilon}$; $\|A\|^{\leftarrow} = \langle \varnothing, A\rangle^{\Upsilon}$.

Proof. Obvious. □

Theorem 1. $(\langle X,Y\rangle \sqcup \langle X',Y'\rangle)^{\Upsilon} = (\langle X,Y\rangle^{\Upsilon\Upsilon} \sqcup \langle X',Y'\rangle^{\Upsilon\Upsilon})^{\Upsilon}$.

Proof. As $(\langle X,Y\rangle \sqcup \langle X',Y'\rangle) \subseteq (\langle X,Y\rangle^{\Upsilon\Upsilon} \sqcup \langle X',Y'\rangle^{\Upsilon\Upsilon})$, $(\langle X,Y\rangle^{\Upsilon\Upsilon} \sqcup \langle X',Y'\rangle^{\Upsilon\Upsilon})^{\Upsilon} \subseteq (\langle X,Y\rangle \sqcup \langle X',Y'\rangle)^{\Upsilon}$ and it is enough to prove the inverse inclusion.

According to the definition,

$(\langle X,Y\rangle \sqcup \langle X',Y'\rangle)^{\Upsilon} = \{\langle X^*,Y^*\rangle \mid X^*,X,X' \vdash Y^*,Y,Y')\}$.

Using Claims 3 and 4 it is further the case that

$= \{\langle X^*,Y^*\rangle \mid \forall X_1 Y_1 \text{ (if } \langle X,Y\rangle \models \langle X_1,Y_1\rangle \text{ then } X^*,X_1,X \vdash Y^*,Y_1,Y)\}$

and using them once more

$= \{\langle X^*,Y^*\rangle \mid \forall X_1 Y_1 X_2 Y_2 \text{ (if } \langle X,Y\rangle \models \langle X_1,Y_1\rangle \text{ and } \langle X,Y'\rangle \models \langle X_2,Y_2\rangle \text{ then } X^*,X_1,X_2 \vdash Y^*,Y_1,Y_2)\}$

which gives us, according to the definitions,

$= (\langle X,Y\rangle^{\Upsilon\Upsilon} \sqcup \langle X',Y'\rangle^{\Upsilon\Upsilon})^{\Upsilon}$. □

Now we can prove

Theorem 2. $\|A\wedge B\|^{\rightarrow} = ((\|A\|^{\rightarrow})^{\Upsilon} \sqcup (\|B\|^{\rightarrow})^{\Upsilon})^{\Upsilon}$.

Proof. According to the definition

$\|A\wedge B\|^{\rightarrow} = \langle\{A,B\}, \emptyset\rangle^{\Upsilon}$,

hence

$\|A\wedge B\|^{\rightarrow} = (\langle A, \emptyset\rangle \sqcup \langle B, \emptyset\rangle)^{\Upsilon}$.

This, according to Theorem 1,

$\|A\wedge B\|^{\rightarrow} = (\langle A, \emptyset\rangle^{\Upsilon\Upsilon} \sqcup \langle B, \emptyset\rangle^{\Upsilon\Upsilon})^{\Upsilon}$

which is nothing else than

$\|A\wedge B\|^{\rightarrow} = ((\|A\|^{\rightarrow})^{\Upsilon} \sqcup (\|B\|^{\rightarrow})^{\Upsilon})^{\Upsilon}$.

In this way we can see, at least partly, into Kaplan's result. But it is also possible to explain Kaplan's proposal in less formal terms, which can be even more illuminating.

9. "Semantics" within syntax

Inferentialistic semantics can be seen as an instance of proof-theoretic semantics (Francez, 2015), bypassing model-theoretic notions like satisfaction, interpretation or model. Interestingly, however, some of these notions can be emulated within the framework of inferentialism. The reason is that a pair $\langle X,Y\rangle$ (or $\langle X,A\rangle$ in the single-conclusion case) can be thought of as Janus-faced: on the one hand it can be seen as a linguistic object (potential inference), while on the other hand it can be seen as a situation. And the confrontation of language and situations may be seen as the core of semantics.

Hence let us consider possible informal interpretations of the pair $<X,Y>$. As before, we may see it as a (potential) *inference*. But it can also be seen differently: in the literature devoted to bilateralism (Restall, 2013; Ripley, 2013) such a pair is considered as a *position*: an epistemic standpoint of somebody who accepts all the X's and rejects all the Y's. And, related to this, we can see such a pair as representing a *situation*, in which all the X's are the case and all the Y's are not. And if $<X,Y>$ represents a correct inference, i.e. if $X \vdash Y$, then $<X,Y>$ is an impossible situation and hence an untenable position.

Given this, if $X \cup X' \vdash Y \cup Y'$, then we can see $<X,Y>$ as an inference that holds in the situation $<X',Y'>$ or which is satisfied by the situation. This lets us present our findings more transparently. In particular, we can see U^{Υ} as the set of all situations satisfying every inference from U. The relation $\models$, then, is that of "semantic consequence": its consequent is satisfied by every situation which satisfies its antecedent. Thus, $U^{\Upsilon\Upsilon}$ is the set if all semantic consequences of U. In particular, $(\|A\|^{\rightarrow})^{\Upsilon} = <A, \emptyset>^{\Upsilon\Upsilon}$ is the set of all semantic consequences of $<A, \emptyset>$. $<X,Y> \sqcup <X',Y'>$, then, is a "fusion" of inferences or situations.

Such "semantic" picturing can sometimes usefully enlighten the subject matter. Take for example an account for the composition of $\|A \wedge B\|^{\rightarrow}$. What we need in order to account for $\|A \wedge B\|^{\rightarrow}$ is to specify the situations in which $<\{A,B\}, \emptyset>$ holds (without referring to A or B). These are the situations in which $<\{A\}, \emptyset> \sqcup <\{B\}, \emptyset>$ holds. We saw that these are the situations in which every fusion of a consequence of $<\{A\}, \emptyset>$ with a consequence of $<\{B\}, \emptyset>$ holds. But the set of all consequences of $<\{A\}, \emptyset>$, $<\{A\}, \emptyset>^{\Upsilon\Upsilon}$, is $((\|A\|^{\rightarrow})^{\Upsilon}$ and similarly for $<\{B\}, \emptyset>$. As a result, $\|A \wedge B\|^{\rightarrow} = <\{A,B\}, \emptyset>^{\Upsilon} = ((\|A\|^{\rightarrow})^{\Upsilon} \sqcup (\|B\|^{\rightarrow})^{\Upsilon})^{\Upsilon}$.

10. Back to single-conclusion inference

Hence we have a solution to the problem of compositionality of inferential role semantics for multiple-conclusion inference and classical logic. But what if we insist on single-conclusion inference? Are the inferential roles instituted by this calculus - and hence presumably the rules for intuitionistic logic[10] - also compositional?

Let us start with disjunction. As we already know that

$$\|A \vee B\|^{\rightarrow} = \|A\|^{\rightarrow} \cap \|B\|^{\rightarrow};$$

[10] See Peregrin (2008).

what we need is $\|A\vee B\|^{\leftarrow}$. Hence consider an X such that

$X \vdash A\vee B$.

Now it is clear that this holds if

for every C such that $A\vee B \vdash C$, $X \vdash C$

and with the help of (CUT) we can show that the former holds only if the latter also holds.

Thus if we return structural rules into the game, we have

$X \vdash A\vee B$ iff for every C such that $\langle\emptyset,C\rangle\in\|A\vee B\|^{\rightarrow}$, $X \vdash C$;

and as $\|A\vee B\|^{\rightarrow} = \|A\|^{\rightarrow}\cap\|B\|^{\rightarrow}$, this holds iff

for every C such that $\langle\emptyset,C\rangle\in\|A\|^{\rightarrow}\cap\|B\|^{\rightarrow}$, $X \vdash C$.

Hence

$\|A\vee B\|^{\leftarrow} = \{X \mid \forall C\text{: if } \langle\emptyset,C\rangle\in\|A\|^{\rightarrow}\cap\|B\|^{\rightarrow} \text{ then } X \vdash C\}$.

Let us move to implication. Again, we know that

$\|A\rightarrow B\|^{\leftarrow} = \{X \mid \exists C\text{: } C\in i(\|B\|) \text{ a } \langle X,C\rangle\in\|A\|^{\rightarrow}\}$

and we need to find out about $\|A\rightarrow B\|^{\rightarrow}$. Let us introduce an auxiliary notation:

$Y \Vdash X \;\equiv_{Def}$ for every $A\in X$, $Y \vdash A$.

Then

$X\cup\{A\rightarrow B\} \vdash C$

is equivalent to

for every Y, if $Y \Vdash X$ and $Y \vdash A\rightarrow B$, then $Y \vdash C$.

Hence

$\|A\rightarrow B\|^{\rightarrow} = \{\langle X,C\rangle \mid \forall Y \text{ if } (Y \Vdash X) \text{ and } (\exists D\text{: } D\in i(\|B\|) \text{ such that } \langle Y,D\rangle\in\|A\|^{\rightarrow}), \text{ then } Y \vdash C\}$.

Now we can define $\neg A$ as $A\rightarrow\bot$, and hence we have

$\|\neg A\|^{\rightarrow} = \{\langle X,C\rangle \mid \forall Y\text{: if } (Y \Vdash X) \text{ and } (\langle Y, \bot\rangle\in\|A\|^{\rightarrow}) \text{ then } \vdash C\}$

$\|\neg A\|^{\leftarrow} = \{X \mid \langle X, \bot\rangle\in\|A\|^{\rightarrow}\}$

If we want to eliminate $\bot$, we can rewrite this as

$\|\neg A\|^{\rightarrow} = \{\langle X,C\rangle \mid \forall Y\text{: if } (Y \Vdash X) \text{ and } (\forall B\text{: } \langle Y,B\rangle\in\|A\|^{\rightarrow}) \text{ then } \vdash C\}$

$$\|\neg A\|^{\leftarrow} = \{X \mid \forall B: \langle X,B\rangle \in \|A\|^{\rightarrow}\}$$

This is no longer the transparently classical negation; and our intuition that the single-conclusion inference lead us to intuitionistic logic is not immediately frustrated. Additionally, we can see that even in the context of single-conclusion inference, the inferential role of a logically complex expression is always determined by the inferential roles of its parts, i.e. inferential roles are compositional.

There is indeed another question concerning how far we are able to *compute* the inferential role of a logically complex expression from those of its parts. Here we must admit that the articulations of the determination we have given are mostly not really useful - in view of the fact that they contain quantification over infinite domains.

11. Conclusion

The question of compositionality of inferential roles of the constraints of propositional logic is surprisingly non-trivial. However, these roles do turn out to be compositional, both in the case of multiple-conclusion inference, and in the case of single-conclusion inference.

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